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HOMOLOGY COBORDISMS, LINK CONCORDANCES, AND HYPERBOLIC 3-MANIFOLDS

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ABSTRACT. Let M_0^3 and M_1^3 be compact, oriented 3-manifolds. They are *homology cobordant* (respectively *relative homology cobordant*) if $\partial M_i^3 = \varnothing$ (resp. $\partial M_i^3 \neq \varnothing$) and there is a smooth, compact oriented 4-manifold W^4 such that $\partial W^4 = M_0^3 - M_1^3$ (resp. $\partial W^4 = M_0^3 - M_1^3) \cup (M_i^3 \times [0,1])$ and $H_*(M_i^3; \mathbf{Z}) \to H_*(W^4; \mathbf{Z})$ are isomorphisms, $i = 0, 1$.

THEOREM. *Every closed, oriented* 3-*manifold is homology cobordant to a hyperbolic* 3-*manifold.*

THEOREM. *Every compact, oriented* 3-*manifold whose boundary is nonempty and contains no* 2-*spheres is relative homology cobordant to a hyperbolic* 3-*manifold.*

Two oriented links L_0 and L_1 in a 3-manifold M^3 are *concordant* if there is a set A^2 of smooth, disjoint, oriented annuli in $M \times [0,1]$ such that $\partial A^2 = L_0 - L_1$, where $L_i \subseteq M^3 \times \{i\}$, $i = 0, 1$.

THEOREM. *Every link in a compact, oriented* 3-*manifold* M^3 *whose boundary contains no* 2-*spheres is concordant to a link whose exterior is hyperbolic.*

COROLLARY. *Every knot in* S^3 *is concordant to a knot whose exterior is hyperbolic.*

1. Introduction. Two closed, oriented 3-manifolds M_0^3 and M_1^3 are *homology cobordant* if there is a compact, oriented 4-manifold W^4 such that $\partial W^4 = M_0^3 - M_1^3$ and the inclusion induced homomorphisms $H_*(M_i^3; \mathbf{Z}) \to H_*(W^4; \mathbf{Z})$ are isomorphisms. The set of homology cobordism classes of homology 3-spheres forms an abelian group θ_3^H, with addition induced by connected sum. At present all that is known of the structure of θ_3^H is the existence of an epimorphism $\mu: \theta_3^H \to \mathbf{Z}_2$, the Rochlin invariant. This group has in recent years become an object of intense study, due to the theorem of Galewski and Stern [**2**] and Matumoto [**9**] that all closed topological n-manifolds, $n \geqslant 6$, admit simplicial triangulations if and only if there exists $\{M^3\} \in \theta_3^H$ such that $\mu\{M^3\} \neq 0$ and $2\{M^3\} = 0$.

It is therefore of some interest to find a restricted set of "nice" 3-manifolds to represent all the homology cobordism classes. The first result in this direction is due to Livingston [**8**], who proved that every closed, oriented 3-manifold is homology cobordant to a Haken manifold. In this paper we further restrict the set of representatives by proving that every closed, oriented 3-manifold is homology cobordant to a hyperbolic 3-manifold (Theorem 5.1). In addition the analogous

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relative result (Theorem 7.1) is proven for compact, oriented 3-manifolds with boundary.

Two oriented links L_0 and L_1 in a 3-manifold M^3 are *concordant* if they can be joined by disjoint smooth annuli in $M^3 \times [0, 1]$, where $L_i \subseteq M \times \{i\}$. The set of concordance classes of knots in S^3 forms the classical knot concordance group $\mathcal{C}_1$, an abelian group with addition induced by composition. Although much more is known about $\mathcal{C}_1$ than about θ_3^H its structure has not yet been completely determined. Thus it may be desirable to find an analogous set of nice representatives. Kirby and Lickorish [**7**] and Livingston [**8**] have shown that every knot in S^3 is concordant to a prime knot. In this paper we show that every link in a compact, oriented 3-manifold whose boundary contains no 2-spheres is concordant to a link whose exterior is hyperbolic (Theorem 7.2).

We shall always deal with hyperbolic 3-manifolds by means of Thurston's Theorem, which asserts that simple Haken manifolds are hyperbolic. It is thus necessary to develop techniques for constructing simple Haken manifolds and homology cobordisms. These matters, along with definitions and notion, are treated in §2. The material in that section as well as others depends heavily on a previous paper [**11**] in which the author proved that a compact, connected, orientable 3-manifold whose boundary contains no 2-spheres contains a knot with (simple Haken) hyperbolic exterior. Using this result we give in §3 a short proof of Livingston's theorem as well as a generalization to manifolds with boundary.

In §4 we prove the existence of a disjoint set of properly embedded arcs in a 3-cell (an "n-tangle") having a (simple Haken) hyperbolic exterior. This result and the notion of concordant n-tangles form the two key ideas in the proof of Theorem 5.1 which is given in §5.

In §6 the main result of [**11**] is modified to show that every compact, connected, orientable 3-manifold whose boundary contains no 2-spheres contains a properly embedded arc whose exterior is (simple Haken) hyperbolic. This is the principal fact needed to establish the results on relative homology cobordisms (Theorem 7.1) and link concordances (Theorem 7.2) given in §7.

The reader should be warned that the proofs of Propositions 4.1 and 6.1, which establish the existence of atoroidal tangles and tunnels and occupy all of §§4 and 6, are quite long and technical. It may be advisible to skip these proofs on a first reading in order to concentrate on the applications of these results to homology cobordisms and link concordances.

2. Preliminaries. We shall work throughout in either the PL or smooth category; all manifolds are orientable and are assumed compact unless otherwise indicated. Submanifolds are assumed to be locally flat PL or smooth and, unless the contrary is evident, to be properly embedded in or contained in the boundary of their ambient manifolds. The *exterior* of a subcomplex or submanifold is the closure of the complement of a regular neighborhood.

A *knot* K in a 3-manifold M is a simple closed curve in the interior of M. A *link* L is a finite disjoint set of knots in M.

The reader is referred to [**4**, **5**, **6** and **14**] for the definitions of incompressible and boundary-imcompressible surfaces, compressing and boundary-compressing disks, parallel surfaces, boundary-parallel surfaces, and parallelisms, and of irreducible, boundary-irreducible, and sufficiently large 3-manifolds. The expressions "surface in a 3-manifold" and "surface in the boundary of a 3-manifold" are used as in [**14**].

A 3-*manifold pair* (M, F) consists of a 3-manifold M and a surface F in ∂M. (M, F) is *irreducible* if M is irreducible and F is incompressible in M. A connected surface G in M with ∂G in F is *F-compressible* if there is a boundary-compressing disk D for G with ∂D in $F \cup G$. G is *F-parallel* if it is parallel to a surface in F. The proofs of the following two lemmas are left to the reader.

2.1 LEMMA. *Let (M, F) be an irreducible* 3-*manifold pair. Then every F-compressible annulus in M is F-parallel.*

2.2 LEMMA. *Let (M_0, F) be an irreducible* 3-*manifold pair and M_1 an irreducible* 3-*manifold such that $M_0 \cap M_1 = \partial M_0 \cap \partial M_1$ is a disjoint set of disks missing F. Then $(M_0 \cup M_1, F)$ is an irreducible* 3-*manifold pair.*

A compact, orientable, irreducible, boundary-irreducible, sufficiently large 3-manifold is called a *Haken manifold*. A compact, orientable, irreducible, boundary-irreducible 3-manifold M is called *simple* if every incompressible annulus and torus in M is boundary-parallel.

A compact 3-manifold M is *hyperbolic* if the complement of the torus components of ∂M has a complete Riemannian metric with finite volume and constant negative sectional curvature with respect to which the nontorus components of ∂M are totally geodesic.

We shall always deal with hyperbolic 3-manifolds through the medium of

2.3 THE MONSTER (THURSTON'S THEOREM). *Simple Haken manifolds are hyperbolic.*

We shall need some sufficient conditions under which the union of two 3-manifolds along incompressible surfaces in their boundaries is simple and Haken. The following definitions are taken from §3 of [**11**], in which the reader can find proofs of the next two lemmas. Let (M, F) be a 3-manifold pair.

(M, F) has *Property A* if

(1) (M, F) and $(M, \overline{\partial M - F})$ are irreducible 3-manifold pairs,

(2) no component of F is a disk or 2-sphere, and

(3) every disk D in M with $D \cap F$ a single arc is boundary-parallel.

(M, F) has *Property B′* if

(1) (M, F) has *Property A*,

(2) no component of F is an annulus or torus,

(3) every incompressible annulus A in M with $\partial A \cap \partial F = \varnothing$ is boundary-parallel, and

(4) every incompressible torus in M is boundary-parallel.

(M, F) has *Property C′* if

(1) (M, F) has Property B′, and

(2) every disk D in M with $D \cap F$ a pair of disjoint arcs is boundary-parallel.

Now suppose $M = M_0 \cup M_1$, where M_0 and M_1 are compact, orientable 3-manifolds and $F = M_0 \cap M_1 = \partial M_0 \cap \partial M_1$ is a compact 2-manifold.

2.4 LEMMA. *If (M_0, F) and (M_1, F) have Property* A, *then M is Haken. In particular, if M_0 and M_1 are Haken, F, $\overline{(\partial M_0 - F)}$ and $\overline{(\partial M_1 - F)}$ are incompressible, and F has no disk or 2-sphere components, then M is Haken.*

2.5 LEMMA. *If (M_0, F) has Property* B′ *and (M_1, F) has Property* C′, *then M is simple and Haken. In particular, if M_0 and M_1 are simple Haken manifolds, F, $\overline{(\partial M_0 - F)}$ and $\overline{(\partial M_1 - F)}$ are incompressible, and no component of F is a disk, 2-sphere, annulus, or torus, then M is a simple Haken manifold.*

A *homology cobordism* is a triple $(W^4; M_0^3, M_1^3)$, where M_0^3 and M_1^3 are closed, oriented 3-manifolds, and W^4 is a compact, oriented 4-manifold such that $\partial W^4 = M_0^3 - M_1^3$ and the inclusion induced homomorphisms $H_*(M_i^3; \mathbf{Z}) \to H_*(W^4; \mathbf{Z})$, $i = 0, 1$, are isomorphisms. A *relative homology cobordism* is a triple $(W^4; M_0^3, M_1^3)$ where M_0^3 and M_1^3 are compact, oriented 3-manifolds with nonempty, homeomorphic boundaries, and W^4 is a compact, oriented 4-manifold such that $\partial W^4 = (M_0^3 - M_1^3) \cup (\partial M_0^3 \times [0, 1])$ and the inclusion induced homomorphisms $H_*(M_i^3; \mathbf{Z}) \to H_*(W^4; \mathbf{Z})$, $i = 0, 1$, are isomorphisms.

Suppose $(P^4; X_0^3, X_1^3)$ and $(Q^4; Y_0^3, Y_1^3)$ are relative homology cobordisms. Let F^2 and G^2 be surfaces in ∂X_0^3 and ∂Y_0^3, respectively. Suppose $\phi\colon F^2 \to G^2$ is a homeomorphism. Let $W^4 = P^4 \cup_{\phi \times \mathrm{id}} Q^4$, $M_0^3 = X_0^3 \cup_{\phi \times \{0\}} Y_0^3$, and $M_1^3 = X_1^3 \cup_{\phi \times \{1\}} Y_1^3$. We denote this construction by

$$(W^4; M_0^3, M_1^3) = (P^4; X_0^3, X_1^3) \cup_\phi (Q^4; Y_0^3, Y_1^3).$$

2.6 LEMMA. *If $F^2 = \partial X_0^3$, then $(W^4; M_0^3, M_1^3)$ is a homology cobordism. If $F^2 \neq \partial X_0^3$, then $(W^4; M_0^3, M_1^3)$ is a relative homology cobordism.*

PROOF. Consider, for $i = 0, 1$, the commutative diagram

$$\begin{array}{ccccccc}
\to & H_j(X_i^3) \oplus H_j(Y_i^3) & \to & H_j(M_i^3) & \to & H_{j-1}(F^2 \times \{i\}) & \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
\to & H_j(P^4) \oplus H_j(Q^4) & \to & H_j(W^4) & \to & H_{j-1}(F^2 \times [0,1]) & \to
\end{array}$$

where the horizontal rows are Mayer-Vietoris sequences and the vertical homomorphisms are inclusion induced. The first and third of these are clearly isomorphisms. Thus by the Five Lemma so is the second.

Now let G_0 and G_1 be homeomorphic 1-complexes in a compact, oriented 3-manifold M^3 such that $G_0 \cap \partial M^3 = G_1 \cap \partial M^3$ consists of vertices. A *concordance* between G_0 and G_1 is a triple $(A^2; G_0, G_1)$ where A^2 is homeomorphic to $G_0 \times [0, 1]$ and is embedded in $M^3 \times [0, 1]$ so that $A^2 \cap (M^3 \times \{i\}) = G_i$, for $i = 0, 1$, and $A^2 \cap (\partial M^3 \times [0, 1]) = (G_0 \cap \partial M^3) \times [0, 1]$. (The embedding is required to be locally flat PL or smooth. The meaning of this is clear if G_0 is a 1-manifold. If G_0 is not a 1-manifold, one can require that the embedding be a product embedding on a neighborhood of the nonmanifold set.) The *exterior* of $(A^2; G_0, G_1)$ is the triple

$(P^4; X_0^3, X_1^3)$ consisting of the exteriors of A^2, G_0, and G_1 in $M^3 \times [0, 1]$, $M^3 \times \{0\}$, and $M^3 \times \{1\}$, respectively.

2.7 LEMMA. *$(P^4; X_0^3, X_1^3)$ is a relative homology cobordism.*

PROOF. The proof is a Mayer-Vietoris sequence and Five Lemma argument similar to that of Lemma 2.6.

3. Homology cobordisms to Haken manifolds.

3.1 THEOREM (LIVINGSTON [8]). *Every closed, oriented 3-manifold M_0^3 is homology cobordant to a Haken manifold M_1^3.*

PROOF. We may assume that M_0^3 is connected. M_0^3 contains a knot J whose exterior X_0^3 is a Haken manifold. The basic idea behind this fact is due to Bing [1]. A proof of this assertion in its present form can be found in González-Acuña [3], Myers [10, 11], or Row [13].

Let K_1 be a nontrivial slice knot in S^3. Then there is a concordance $(A^2; K_0, K_1)$ in $(S^3 \times I; S^3 \times \{0\}, S^3 \times \{1\})$, where K_0 is the trivial knot. By Lemma 2.7 the exterior $(Q^4; Y_0^3, Y_1^3)$ of this concordance is a relative homology cobordism. Y_1^3 is Haken and Y_0^3 is a solid torus. Let ϕ: $\partial X_0^3 \to \partial Y_0^3$ be a homeomorphism which extends to a homeomorphism from $(\overline{M_0^3 - X_0^3})$ to Y_0^3. Let $(P^4; X_0^3, X_1^3) = (X_0^3 \times [0, 1]; X_0^3 \times \{0\}, X_0^3 \times \{1\})$ be a product relative homology cobordism. Then by Lemma 2.6 $(W^4; M_0^3, M_1^3) = (P^4; X_0^3, X_1^3) \cup_\phi (Q^4; Y_0^3, Y_1^3)$ is a homology cobordism. By Lemma 2.4 $M_1^3 \cup_\phi Y_1^3$ is Haken.

3.2 THEOREM. *Every compact, oriented 3-manifold M_0^3 such that $\partial M_0^3 \neq \varnothing$ and contains no 2-spheres is relative homology cobordant to a Haken manifold M_1^3.*

PROOF. We may assume that M_0^3 is connected. In [11] the author proved that every compact, connected, oriented 3-manifold whose boundary contains no 2-spheres (and may be empty) contains a knot J whose exterior is Haken (in fact simple Haken). The remainder of the proof is similar to that of the previous theorem.

4. Atoroidal n-tangles. An *n-tangle* is a set $\{\lambda_1, \ldots, \lambda_n\}$ of disjoint, properly embedded arcs in a 3-cell B. An *n-tangle space* is the exterior of an n-tangle in B. An n-tangle is *atoroidal* if its n-tangle space is simple.

In [11] the author considered the atoroidal 2-tangle in Figure 1, called the true lover's tangle. The n-tangle in Figure 2, illustrated for $n = 3$, is a generalization of the true lover's tangle called the true lover's n-tangle.

4.1 PROPOSITION. *The true lover's n-tangle is atoroidal.*

PROOF. By [11] we may assume that $n \geqslant 3$. The true lover's n-tangle space can be expressed as the union of $2n - 1$ cubes with handles, as illustrated in Figure 3. The proof consists in showing that $(\bigcup_{j=1}^{n} P_{2j-1}, \bigcup_{k=1}^{2n} F_k)$ has Property B′ and $(\bigcup_{j=1}^{n-1} P_{2j}, \bigcup_{k=1}^{2n} F_k)$ has Property C′. It is sufficient to prove that (P_1, F_1) has Property B′, $(P_2, F_1 \cup F_2)$ has Property C′, and $(P_3, F_2 \cup F_3)$ has Property B′. The first two of these facts were proved in Lemmas 4.6 and 4.9 of [11] (where P_2 was called P).

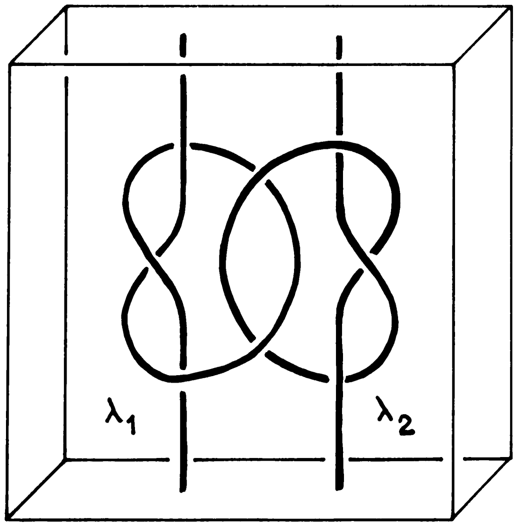

FIGURE 1

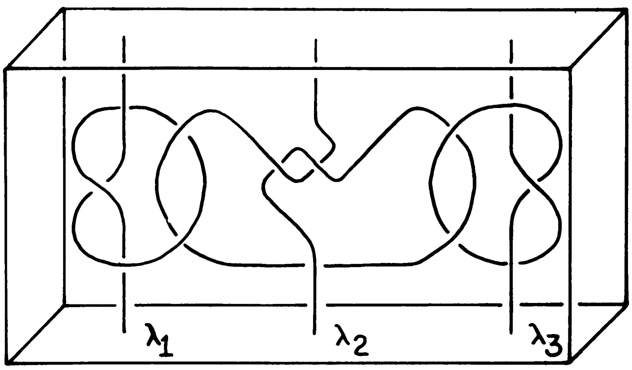

FIGURE 2

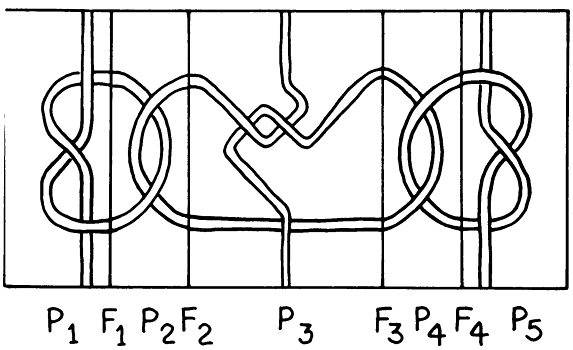

FIGURE 3

∂P_3 is the union of the planar surfaces F_2, F_3, and G_3, and the annuli U_a, U_b, and U_c shown in Figure 4. $\pi_1(P_3)$ is free on a, b and c, $\pi_1(F_2)$ is free on a and b, $\pi_1(F_3)$ is free on b' and c', and $\pi_1(G_3)$ is free on a', c, and d. The following relations hold: $a' = acac^{-1}a^{-1}$, $d = ab$, $b' = c^{-1}bc$, $c' = aca^{-1}$, and $d' = aca^{-1}c^{-1}bc$.

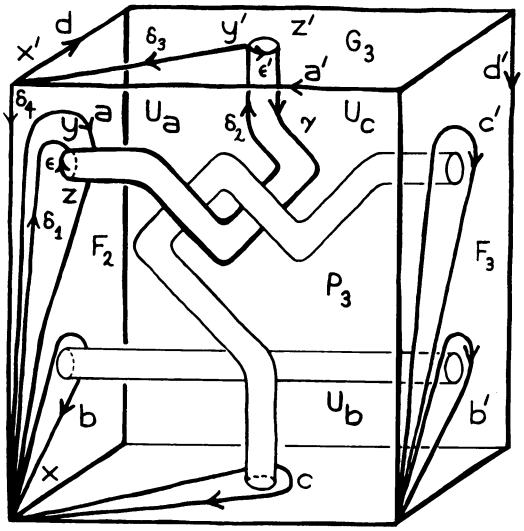

FIGURE 4

In the following algebraic lemmas W' denotes a word in a', c, and d, and W a word in a, b, and c. $V \equiv W$ means that the words V and W are identical, while $V = W$ means that they determine the same element of $\pi_1(P_3)$. Every nontrivial reduced word W' determines a nontrivial reduced word W, which is called a *G-word*. Thus G_3 is incompressible in P_3. Let p and r be elements of $\{a, b^{-1}\}$. Words of the form $pca^mc^{-1}r^{-1}$ or pr^{-1} are called *special words*. Every G-word is a positive power product of c, c^{-1}, and special word. (Note that although every such product is an element of $\pi_1(G_3)$, not every such product is a G-word, since certain juxtapositions result in nonreduced words.)

Let S, P, Q, R be, respectively, a special word, an initial segment of a special word, a G-word, and a terminal segment of a special word. S is never a (proper) segment of a special word. One never has $P \equiv R$. Since S does not begin or end in c^k, this implies that P is never a terminal segment of Q, and R is never an initial segment of Q. Moreover, if $P \not\equiv a$, then P is never a noninitial segment of S, and if $R \not\equiv a^{-1}$, R is never a nonterminal segment of S. It follows that:

(1) If XQZ is a G-word, then X and Z are G-words.

(2) If XPZ is a G-word, and $P \not\equiv a$, then X and PZ are G-words.

(3) If XRZ is a G-word, and $R \not\equiv a^{-1}$, then XR and Z are G-words.

From these facts it follows that the above factorization of a G-word is unique.

4.2 LEMMA. $\pi_1(F_2) \cap \pi_1(G_3) = gp(d)$.

PROOF. Clearly $gp(d) \subseteq (\pi_1(F_2) \cap \pi_1(G_3))$. Suppose $w \in (\pi_1(F_2) \cap \pi_1(G_3))$ is represented by the G-word W. Since $w \in \pi_1(F_2)$, W does not involve c, hence W does not involve $c^{\pm 1}$ or $pca^mc^{-1}r^{-1}$ and is therefore a power of d.

4.3 LEMMA. $\pi_1(F_2) \cap \pi_1(G_3)aca^{-1} = \varnothing$.

PROOF. If $v \in \pi_1(F_2) \cap \pi_1(G_3)aca^{-1}$, then $v = waca^{-1}$ where $w \in \pi_1(G_3)$ and is represented by the G-word W. Since $v \in \pi_1(F_2)$, $Waca^{-1}$ is not reduced, hence W terminates in $pca^mc^{-1}a^{-1}$ or $b^{-1}a^{-1}$. Therefore the reduction of $Waca^{-1}$ is W_0pca^{m-1} or $W_0b^{-1}c^{-1}a^{-1}$, which cannot be in $\pi_1(F_2)$, a contradiction.

4.4 LEMMA. $\pi_1(F_2) \cap ac^{-1}a^{-1}\pi_1(G_3)aca^{-1} = gp(a)$.

PROOF. $gp(a) \subseteq (\pi_1(F_2) \cap ac^{-1}a^{-1}\pi_1(G_3)aca^{-1})$ because

$$a^m = (ac^{-1}a^{-1})(aca^mc^{-1}a^{-1})(aca^{-1}).$$

Suppose $v \in (\pi_1(F_2) \cap ac^{-1}a^{-1}\pi_1(G_3)aca^{-1})$. Then $v = ac^{-1}a^{-1}waca^{-1}$, where $w \in \pi_1(G_3)$ is represented by the G-word W. Let V represent v. Then $aca^{-1}Vac^{-1}a^{-1} = W$. W is the reduction of $aca^{-1}Vac^{-1}a^{-1}$. If $V \not\equiv a^m$, then, since V does not involve c, $W \equiv acV_0c^{-1}a^{-1}$, where V_0 is the reduction of $a^{-1}Va$. W begins and ends in words of the form $pca^mc^{-1}r^{-1}$, which forces V_0 and hence V to involve c, a contradiction.

4.5 LEMMA. *If $v \in \pi_1(P_3)$ and $v^n \in \pi_1(G_3)$, $n \neq 0$, then $v \in \pi_1(G_3)$.*

PROOF. We may assume that $v \neq 1$ and $n > 1$. Let V and W be reduced words representing, respectively, v and v^n. Let S, P, Q, R be as above.

Suppose V^n is reduced. If V is a G-word, then $v \in \pi_1(G_3)$. If V is not a G-word, then $V \equiv P$ or QP, and so $V^n \equiv P^n$ or $(QP)^n$, both of which are impossible.

Suppose V^n is not reduced. Then $V \equiv XY \equiv ZX^{-1}$. If $l(X) \geqslant \frac{1}{2}l(V)$, then $V \equiv ZMY$, where $X \equiv ZM$ and $X^{-1} \equiv MY$. Since $X^{-1} \equiv M^{-1}Z^{-1}$ and $l(M) = l(M)^{-1}$, $M \equiv M^{-1}$, so that $M^2 = 1$. This implies that $M = 1$ because $\pi_1(P_3)$ is free. Therefore $V \equiv ZY \equiv XX^{-1} \equiv 1$, a contradiction. Hence $l(X) < \frac{1}{2}l(V)$ and so $V \equiv XYX^{-1}$ and $W \equiv XY^nX^{-1}$, which is reduced. Regard Y^n as $Y_1Y_2 \cdots Y_n$, where each $Y_i \equiv Y$. In the hypotheses of the following cases P and R refer, respectively, to initial and terminal segments of special words embedded in W and overlapping with Y_1 as indicated.

Case 1. $Y_1 \equiv Q$. Then Y^n is a G-word, and so X and X^{-1} are G-words. Therefore $V \equiv XQX^{-1}$ is a G-word.

Case 2. $Y_1 \equiv P$. Then X is a G-word. Hence X^{-1} and Y^n are G-words. By the argument for V^n reduced, Y is a G-word, contradicting $Y_1 \equiv P$.

Case 3. $Y_1 \equiv R$. Then XY_1 is a G-word, and so $Y_2 \cdots Y_nX^{-1}$ is a G-word, which is impossible since $Y_2 \equiv R$.

Case 4. Y_1 is a noninitial, nonterminal segment of S. Then $S \equiv pca^mc^{-1}r^{-1}$. $Y_2 \equiv Y_1$ implies that $Y_1 \equiv a^k$. W reduced implies that $X \equiv X_0pc$, where X_0 is a G-word. Hence $V \equiv X_0pca^kc^{-1}p^{-1}X_0^{-1}$, which is a G-word.

Case 5. $Y_1 \equiv RQ$. Then XRQ is a G-word. So $(RQ)^{n-1}X^{-1}$ is a G-word, which is impossible.

Case 6. $Y_1 \equiv QP$. Then X is a G-word. Hence X^{-1} is a G-word, and so $P(QP)^{n-1}$ is a G-word, which is impossible.

Case 7. $Y_1 \equiv RQP$. Then $W \equiv XR(QRP)^{n-1}QPX^{-1}$ implies that XR, RP and PX^{-1} are G-words. Thus $V \equiv XRQPX^{-1}$ is a G-word.

Case 8. $Y_1 \equiv RP$. Then $W \equiv X(RP)^n X^{-1}$. XR is a G-word. If $P \not\equiv a$, then PX^{-1} is a G-word. If $P \equiv a$, then $R \not\equiv a^{-1}$. So $X(RP)^{n-1}R$ and PX^{-1} are G-words. Thus $V \equiv XRPX^{-1}$ is a G-word.

This completes the proof.

In the following lemmas, J_a, $J_{a'}$, etc. denote the component of ∂F_2, ∂G_3, etc. homologous to a, a', etc. f and g denote elements of $\pi_1(F_2)$ and $\pi_1(G_3)$, respectively. Let $F = F_2 \cup F_3$.

4.6 LEMMA. *(P_3, F) has Property* A.

PROOF. P_3 is a cube with handles and is therefore irreducible. No component of F is a disk or 2-sphere. F is incompressible for homological reasons. Thus U_a, U_b, and U_c are incompressible. As noted above, G_3 is incompressible.

Let $\gamma, \delta_1, \delta_2, \delta_3, \delta_4, \varepsilon$ and ε' be the arcs and x, x', y, y' and z be the points in Figure 4. Let $\delta = \delta_1\delta_2\delta_3\delta_4$. Then $[\delta] = ac^{-1}a^{-1}$, and $\varepsilon'\gamma\varepsilon^{-1}$ is homotopic to δ_2^{-1} rel$\{y, y'\}$.

Let D be a disk in P_3 with $\partial D \cap F$ a single arc α. Let $\beta = \overline{\partial D - \alpha}$. Note that there is a homeomorphism of P_3 which interchanges F_2 and F_3, interchanges U_a and U_c, and leaves U_b invariant. We may thus assume that $\alpha \subseteq F_2$.

Case 1. $\alpha \cap J_b \neq \varnothing$.. Then β is a J_b-parallel arc in U_b. Hence D can be isotoped so that $\partial D \subseteq F_2$. By the incompressibility of F_2 and irreducibility of P_3, D is boundary-parallel.

Case 2. $\partial\alpha \subseteq J_d$. Isotop D and orient ∂D so that α runs from x to x'. Then $[\partial D] = [\alpha\delta_4][\delta_4^{-1}\beta] = fg = 1$. Hence $f \in \pi_1(F_2) \cap \pi_1(G_3)$. By Lemma 4.2, $f = d^k$. Since $\alpha\delta_4$ is a simple closed curve, $|k| \leqslant 1$. It follows that α is parallel to δ_4 or to $\overline{J_d - \delta_4}$ in F_2. Thus D can be isotoped so that $\partial D \subseteq G_3$. Hence D is boundary-parallel.

Case 3. α runs from J_d to J_a. Isotop D and orient ∂D so that α runs from x to y and $\beta \cap U_a = \delta_2$. Let $\beta' = \beta \cap G_3$. Then

$$[\partial D] = \left[\alpha\delta_1^{-1}\right]\left[\delta_1\delta_2\delta_3\delta_4\right]\left[\delta_4^{-1}\delta_3^{-1}\beta'\right] = fac^{-1}a^{-1}g = 1.$$

So $f = g^{-1}aca^{-1} \in \pi_1(F_2) \cap \pi_1(G_3)aca^{-1}$, contradicting Lemma 4.3. Therefore this case cannot occur.

Case 4. $\partial\alpha \subseteq J_a$. Isotop D and orient ∂D so that α runs from z to y. We may assume that $\beta = \beta_1\beta_2\beta_3$, where $\beta_1 = \delta_2$, β_2 runs from y' to z' in G_3, and $\beta_3 = \gamma$. Referring $[\partial D]$ to x via δ_1, we have

$$\begin{aligned} 1 = [\partial D] &= \left[\delta_1\beta_1\beta_1\beta_3\alpha\delta_1^{-1}\right] \\ &= \left[\delta_1\delta_2\delta_3\delta_4\right]\left[\delta_4^{-1}\delta_3^{-1}\beta_2(\varepsilon')^{-1}\delta_3\delta_4\right]\left[\delta_4^{-1}\delta_3^{-1}(\varepsilon'\gamma\varepsilon^{-1})\delta_1^{-1}\right]\left[\delta_1\varepsilon\alpha\delta_1^{-1}\right] \\ &= [\delta]g[\delta]^{-1}f. \text{ So } f^{-1} = ac^{-1}a^{-1}gaca^{-1}. \end{aligned}$$

By Lemma 4.4 $f = a^m$. Since $\varepsilon\alpha$ is a simple closed curve, $|m| \leqslant 1$ and hence α is parallel to ε or $\overline{J_a - \varepsilon}$ in F_2. Thus D can be isotoped so that $\partial D \subseteq G_3$ and so is boundary-parallel.

4.7 LEMMA. *(P_3, F) has Property* B$'$.

PROOF. Clearly P_3 contains no incompressible tori. Suppose A is an incompressible annulus in P_3 with $\partial A \cap \partial F = \varnothing$. Let α and β be the components of ∂A. We may assume that $\partial A \subseteq (F \cup G_3)$.

Case 1. α and β are parallel in ∂P_3. Then α and β cobound an annulus A' in ∂P_3. A can be isotoped so that α and β lie in the same component of $(P_3 - \partial F)$, which we may assume to be F_2 or G_3. Let T' be the result of isotoping the torus $T = A \cup A'$ slightly into Int P_3. T' is compressible in P_3 and thus bounds either a solid torus V' or knot exterior Q' in P_3. If T' bounds Q', then the boundary of a compressing disk for T' must be a meridian of Q'. Since A' lies on the boundary of an obvious 3-cell containing P_3, α is homotopic in P_3 to a meridian of Q'. This contradicts the incompressibility of A in P_3. Therefore T' bounds V', and hence T bounds a solid torus V. If A is not parallel to A' across V, then the core of V, referred to the basepoint, represents a root of an element of $\pi_1(F_2)$ which does not lie in $\pi_1(G_3)$. The first possibility is ruled out by $\pi_1(F_2)$ being a free factor of $\pi_1(P_3)$. The second is ruled out by Lemma 4.5. Thus A is boundary-parallel.

Case 2. α and β are not parallel in ∂P_3. We show that this cannot happen. For homological reasons, we may assume that α is parallel to J_b in F_2 and β separates $J_{a'} \cup J_d$ from $J_c \cup J_{d'}$ in G_3. Regard the disk E in Figure 5 as $[0,2] \times [0,1]$, where $\{0\} \times [0,1] \subseteq U_a$, $\{2\} \times [0,1] \subseteq U_c$, $[0,1] \times \{0\} \subseteq F_2$, $[1,2] \times \{0\} \subseteq G_3$, $[0,1] \times \{1\} \subseteq G_3$, and $[1,2] \times \{1\} \subseteq F_2$. Let θ be the arc on ∂P_3 running from $x = \{1\} \times \{0\}$ to $w = \{1\} \times \{1\}$ and ω the arc $\{1\} \times [0,1]$ running from w to x shown in Figure 5. $[\theta\omega] = ca$, which is clearly not in $\pi_1(G_3)$.

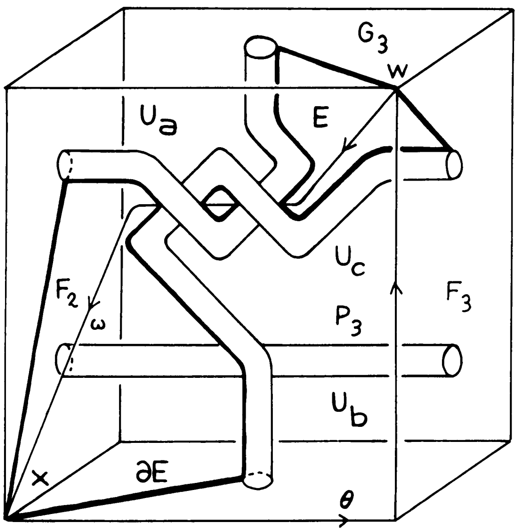

FIGURE 5

Isotop A, keeping α in F_2 and β in G_3, so that A and E are in general position and meet in a minimal number of components. Since $[0,1] \times \{1\}$ joins $J_{a'}$ to $J_{d'}$, $A \cap E \neq \varnothing$.

If each component of $A \cap E$ is parallel in E to an arc in G_3, then there is a component ρ which cobounds a disk D_0 on A with an arc ρ_0 in β and a disk D_1 on E with an arc ρ_1 in G_3 such that $E \cap \operatorname{Int} D_0 = \varnothing$. By the incompressibility of G_3 and irreducibility of P_3, the disk $D_0 \cup D_1$ is G_3-parallel. It follows that A can be isotoped, keeping α in F_2 and β in G_3, so as to remove at least ρ from $A \cap E$, contradicting minimality.

Therefore some component ρ of $A \cap E$ runs from $[0,1] \times \{1\}$ to $[1,2] \times \{0\}$. Let ξ be an arc in $[0,1] \times \{1\}$ running from w to the origin of ρ, and let η be an arc in $[1,2] \times \{0\}$ running from the terminus of ρ to x. $\xi\rho\eta$ is homotopic in E to ω, and ρ is homotopic in A to an arc ρ' in β, both homotopies fixing x and w. Therefore $ca = [\theta\omega] = [\theta\xi\rho\eta] = [\theta\xi\rho'\eta] \in \pi_1(G_3)$, a contradiction. This completes the proof.

5. Homology cobordisms to hyperbolic 3-manifolds.

5.1 THEOREM. *Every closed, oriented 3-manifold M_0^3 is homology cobordant to a hyperbolic 3-manifold M_1^3.*

The proof depends on

5.2 LEMMA. *Every cube with handles X_0^3 (of genus at least one) is relative homology cobordant to a simple Haken (and thus hyperbolic) 3-manifold X_1^3.*

PROOF. If X_0^3 is a solid torus, then let X_1^3 be the exterior of any nontrivial, simple slice knot in S^3, e.g. the stevedore's knot [**12**]. The exterior of a concordance between this knot and the trivial knot is the required homology cobordism.

We therefore assume that X_0^3 has genus $n \geqslant 2$. Let $\{\mu_1, \dots, \mu_n\}$ be the n-tangle in the 3-cell B^3 in Figure 6 with exterior X_1^3. $\{\mu_1, \dots, \mu_n\}$ is the composition of the true lover's n-tangle $\{\lambda_1, \dots, \lambda_n\}$ with its reflection in a plane. As illustrated in Figure 7, $\{\mu_1, \dots, \mu_n\}$ is concordant to a trivial tangle $\{\mu_1', \dots, \mu_n'\}$ whose exterior can be identified with X_0^3. By Lemma 2.7 the exterior $(P^4;\ X_0^3,\ X_1^3)$ of this concordance is a relative homology cobordism.

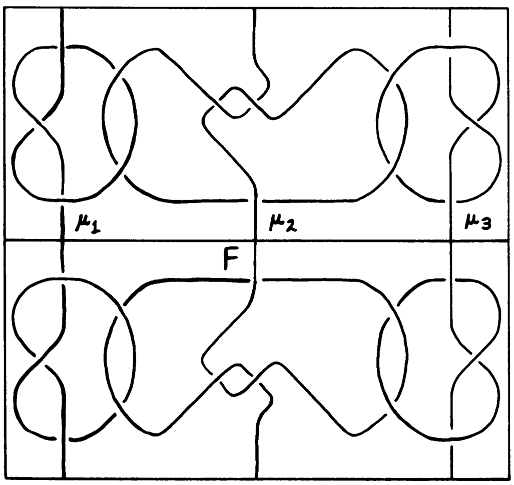

FIGURE 6

$X_1^3 = Q^3 \cup (-Q^3)$, where Q^3 is the true lover's n-tangle space and $Q^3 \cap (-Q^3) = F$ is the planar surface in Figure 6. By Proposition 4.1 Q^3 is simple. Since F is clearly incompressible in Q^3 and $-Q^3$, (Q^3, F) and $(-Q^3, F)$ have Property C′. Since Q^3 is clearly Haken, Lemma 2.5 implies that X_1^3 is simple and Haken.

(a)

(b)

(c)

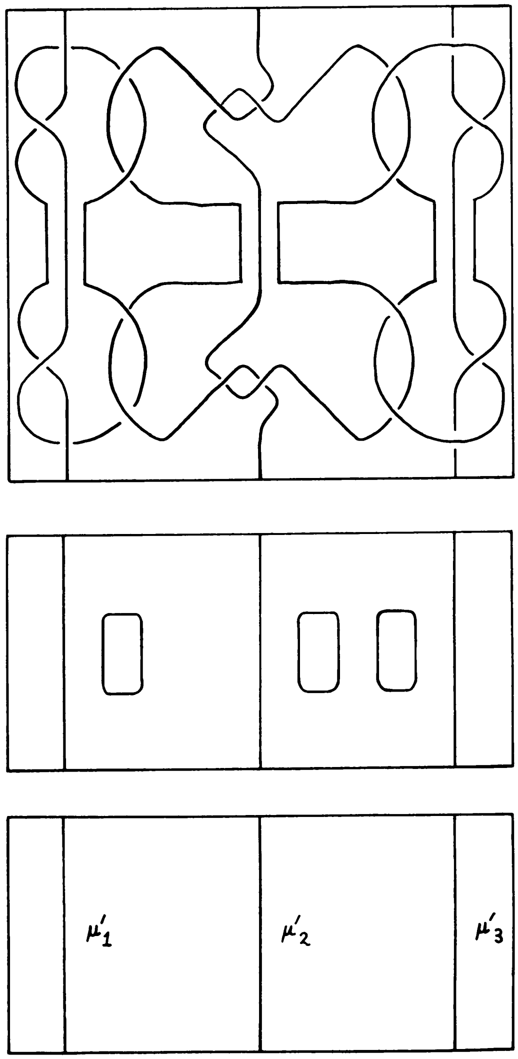

FIGURE 7

PROOF OF THEOREM 5.1. We may assume that M_0^3 is connected. Let $X_0^3 \cup_\phi Y_0^3$ be a Heegaard splitting of M_0^3, i.e. X_0^3 and Y_0^3 are cubes with handles, and $\phi\colon \partial X_0^3 \to \partial Y_0^3$ is a homeomorphism under whose identification one obtains M_0^3. We may assume, by adding trivial handles if necessary, that the splitting has genus at least two. Let $(P^4; X_0^3, X_1^3)$ and $(Q^4; Y_0^3, Y_1^3)$ be relative homology cobordisms, where X_1^3 and Y_1^3 are simple Haken manifolds. Let $(W^4; M_0^3, M_1^3) = (P^4; X_0^3, X_1^3) \cup_\phi (Q^4; Y_0^3, Y_1^3)$. By Lemma 2.6 $(W^4; M_0^3, M_1^3)$ is a homology cobordism. By Lemma 2.5 M_1^3 is simple and Haken and therefore hyperbolic.

6. Atoroidal tunnels.

6.1 PROPOSITION. *Let M be a compact, connected, orientable 3-manifold such that $\partial M \neq \varnothing$ and contains no 2-spheres. Let F be a component of ∂M. Then M contains a properly embedded arc $\tilde{J}$ such that $\partial \tilde{J} \subseteq F$ and the exterior of $\tilde{J}$ is a simple Haken manifold.*

The proof will be a modification of §§5 and 6 of **[11]**, in which the author constructed a knot in M whose exterior is a simple Haken manifold. Let $C = \partial M \times [0, 1]$ be a collar on ∂M, with $\partial M = \partial M \times \{1\}$. Let $N = \overline{(M - C)}$. In Lemma 5.1 of **[11]** it was shown that N admits a *special handle decomposition* $\{h_i^k\}$, i.e. a handle decomposition **[14]** such that:

(1) $h_i^k \cap \partial N$ and $h_i^k \cap h_j^l$, $i \neq j$, are each either empty or connected,

(2) each 0-handle h_i^0 meets exactly four 1-handles and six 2-handles,

(3) each 1-handle h_j^1 meets exactly two 0-handles and three 2-handles,

(4) each pair h_k^2, h_l^2 of 2-handles either

(a) meets no common 0- or 1-handle, or

(b) meets exactly one common 0-handle and no common 1-handle, or

(c) meets exactly one common 1-handle and two common 0-handles,

(5) the complement of any 0-handle in the union H' of the 0- and 1-handles is connected, and

(6) the union of any 0-handle with the union H'' of the 2- and 3-handles is a cube with handles which meets ∂N in a disjoint collection of disks.

Let C_i^k be the core of the handle h_i^k. Let $Z = H'' \cup C$, $R_i = h_i^0 \cap Z$, and $S_j = h_j^1 \cap Z$.

In **[11]** it was shown that $\cup\, C_j^1$, together with atoroidal 2-tangles in the h_i^0, formed a knot J whose exterior is a simple Haken manifold. The arc $\tilde{J}$ will be constructed by deleting one C_0^1 whose 1-handle h_0^1 meets $F \times \{0\}$ and connecting the endpoints to F via product arcs in C. To be more specific, let h_0^1 be a 1-handle which meets $F \times \{0\}$. Let h_0^0, h_1^0, h_0^2, h_1^2, and h_2^2 be the 0- and 2-handles meeting h_0^1 and h_1^1, h_2^1 and h_3^1 the 1-handles meeting h_0^0. We may assume that $(h_0^2 \cup h_3^1) \cap \partial N = \varnothing$. For $i = 0, 1$, let E_i be a disk in $\operatorname{Int}(R_i \cap \partial N)$, $g_i = E_i \times [0, 1]$, $G_i = (\partial E_i) \times [0, 1]$, and $\mu_i = x_i \times [0, 1]$ for $x_i \in \operatorname{Int} E_i$, where the product structure refers to C.

We shall need the following additional properties of our special handle decomposition.

6.2 LEMMA. *With the notation as above*:

(7) *for any* 0-*handle* h_i^0, $H' - h_i^0 - h_0^1$ *is connected*,

(8) $H'' \cup h_0^1$ *is a cube with handles which meets* ∂N *in a disjoint set of disks*,

(9) *for any* 0-*handle* h_i^0, $H'' \cup h_i^0 \cup h_0^1$ *is a cube with handles which meets* ∂N *in a disjoint set of disks*, *and*

(10) $H'' \cup h_0^0 \cup h_0^1 \cup h_1^1 \cup h_2^1 \cup h_3^1$ *is a cube with handles which meets* ∂N *in a disjoint set of disks.*

PROOF. (7) By (5) $H' - h_i^0$ is connected. If $i = 0$ the result is clear. If $i \neq 0$, then there is a 2-handle h_k^2 which meets h_0^1 but does not meet h_i^0. Since $h_k^2 \cap (H' - h_i^0 - h_0^1)$ is connected the result follows.

(8) Since $H'' \cap h_0^1$ is a 2-cell, $H'' \cup h_0^1$ is a cube with handles. $H'' \cap \partial N$ consists of disjoint disks, two of which are $h_1^2 \cap \partial N$ and $h_2^2 \cap \partial N$. $(H'' \cup h_0^1) \cap \partial N$ is obtained from $H'' \cap \partial N$ by joining these two disks by the disk $h_0^1 \cap \partial N$ which meets each of them in a single arc and therefore consists of disks.

(9) By (6) $H'' \cup h_i^0$ is a cube with handles meeting ∂N in a disjoint set of disks. If $h_i^0 \cap \partial N = \varnothing$, then the proof is the same as that of (8). Suppose $h_i^0 \cap \partial N \neq \varnothing$. If $i = 0$ or 1, the result is clear. If $i \neq 0$ or 1, then let h_k^2, h_l^2, and h_m^2 be the 2-handles which meet both h_i^0 and ∂N. $(H'' \cup h_i^0) \cap \partial N$ is formed from $H'' \cap \partial N$ by joining $h_k^2 \cap \partial N$, $h_l^2 \cap \partial N$, and $h_m^2 \cap \partial N$ along the disk $h_i^0 \cap \partial N$, which meets each of them in a single arc, to form a disk D. If D does not meet h_0^1, then the proof is as in (8). At most one of h_k^2, h_l^2, and h_m^2 meets h_0^1. Thus $(H'' \cup h_i^0 \cup h_1^0) \cap \partial N$ is obtained from $(H'' \cup h_i^0) \cap \partial N$ by joining some $h_n^2 \cap \partial N$ to either D or $h_p^2 \cap \partial N$ along the disk $h_0^1 \cap \partial N$ which meets each in exactly one arc. Thus the result follows.

(10) Let $X = h_0^0 \cup h_0^1 \cup h_1^1 \cup h_2^1 \cup h_3^1$. X is a 3-cell such that $X \cap H''$ is an annulus with a centerline the same as a centerline σ of S_3. We may therefore regard X as a 2-handle attached to H'' along σ. The disk C_0^2 in H'' meets σ transversely in a single point. From this it is easy to see that $H'' \cup X$ is a cube with handles having genus one less than that of H''. Clearly $(H'' \cup X) \cap \partial N$ is homeomorphic to $(H'' \cup h_0^0) \cap \partial N$ and so by (6) is a disjoint set of disks. This completes the proof.

Now let $\tilde{F} = (F - E_0 - E_1) \times \{1\}$, $\tilde{C} = \overline{(C - g_0 - g_1)}$, $\tilde{R}_i = \overline{(R_i - E_i)} \cup (R_i \cap h_0^1)$ for $i = 0$, 1, and R_i for $i \geqslant 2$. Let $\tilde{R} = \cup \tilde{R}_i$. Let $\tilde{Z} = \tilde{C} \cup H'' \cup h_0^1$ and $\tilde{S} = \cup S_j - S_0$.

6.3 Lemma. *$\tilde{R}$ is incompressible in $\tilde{Z}$.*

Proof. Suppose D is a compressing disk for $\tilde{R}_i$. $\partial D = \partial D'$ for a disk D' in ∂h_i^0. Let $S = D \cup D'$.

Case 1. *$i = 0$ or 1, say $i = 0$.* $\tilde{Z} \cup h_0^0 = \tilde{C} \cup H'' \cup H'' \cup h_0^0 \cup h_0^1$. By (9) $H'' \cup h_0^0 \cup h_0^1$ is a cube with handles meeting $\tilde{C}$ in a disjoint set of disks in $\tilde{F} \times \{0\}$. By Lemma 2.2 $\tilde{Z} \cup h_0^0$ is irreducible. So S bounds a 3-cell B in $\tilde{Z} \cup h_0^0$. Since by (7) $H' - h_0^0 - h_0^1$ is connected, it must lie in B or $\overline{(M - B)}$. Thus $h_0^0 \cap \overline{(H' - h_0^0 - h_0^1)}$ lies in D' or $\tilde{R}_0 - D'$. So either $D' \subseteq \tilde{R}_0$, and we are done, or ∂D is parallel in $\tilde{R}_0$ to ∂E_0. If the latter is true, then $g_0 \subseteq \overline{(M - B)}$ and so $H' - h_0^0 - h_0^1 \subseteq B$. But this is impossible since $S \cap g_1 = \varnothing$ and g_1 joins $H' - h_0^0$ to ∂M.

Case 2. *$i \neq 0$ or 1.* By (9) $H'' \cup h_i^0 \cup h_0^1$ is a cube with handles meeting $\tilde{C}$ in a disjoint set of disks in $\tilde{F} \times \{0\}$. It follows that $\tilde{Z} \cup h_i^0 = \tilde{C} \cup H'' \cup h_i^0 \cup h_0^1$ is irreducible. So S bounds a 3-cell B in $\tilde{Z} \cup h_i^0$. By (7) $H'' - h_i^0 - h_0^1$ is connected and so must lie in B or $\overline{(M - B)}$. In either case $h_i^0 \cap \overline{(H' - h_0^1 - h_i^0)}$ lies in D' or $\partial h_i^0 - D'$ and so $\partial D = \partial D''$ for some disk D'' in R_i. This completes the proof.

6.4 Lemma. *$\overline{(\partial \tilde{Z} - \tilde{R})}$ is incompressible in $\tilde{Z}$, which is irreducible.*

Proof. The incompressibility of $\tilde{S}$ follows from that of $\tilde{R}$. By (8) $H'' \cup h_0^1$ is a cube with handles meeting $\tilde{C}$ in a disjoint set of disks in $\tilde{F} \times \{0\}$. The result now follows from Lemma 2.2.

6.5 Lemma. *$(\tilde{Z}, \tilde{R})$ has Property* A.

PROOF. Let D be a disk in $\tilde{Z}$ with $\partial D \cap \tilde{R}$ a single arc α. Let $\beta = \overline{(\partial D - \alpha)}$.

Suppose $\alpha \cap (g_0 \cup g_1) = \varnothing$. Then β is a boundary-parallel arc in $\tilde{S}$. So D can be isotoped so that $D \subseteq \tilde{R}$ and is therefore boundary-parallel by the irreducibility of $\tilde{Z}$ and incompressibility of $\tilde{R}$.

Suppose $\alpha \cap (g_0 \cup g_1) \neq \varnothing$. We may assume that $\alpha \subseteq R_0$, $\partial\alpha \subseteq \partial E_0$, and $\beta = \beta_0 \cup \tilde{\beta} \cup \beta_1$, where $\beta_n = y_n \times [0,1]$ for $y_n \in \partial E_0$, $n = 0, 1$, and $\tilde{\beta} \subseteq \tilde{F} \times \{1\}$. If α is boundary-parallel in $\tilde{R}_0$ or β is boundary-parallel in $\tilde{F}$, then D can be isotoped so that ∂D is in, respectively, $\tilde{F}$ or $\tilde{R}_0$. As above, this implies that D is boundary-parallel. So we may assume that neither of these is the case.

By (10) $H'' \cup h_0^0 \cup h_0^1 \cup h_1^1 \cup h_2^1 \cup h_3^1$ is a cube with handles which meets $(\tilde{F} \cup E_0) \times [0,1]$ in a disjoint collection of disks in $(\tilde{F} \cup E_0) \times \{0\}$. Therefore $(\tilde{F} \cup E_0) \times \{1\}$ is incompressible in

$$\begin{aligned}\tilde{Z} \cup g_0 \cup h_0^0 \cup h_1^1 \cup h_2^1 \cup h_3^1 \\ = \big((\tilde{F} \cup E_0) \times [0,1]\big) \cup \big(H'' \cup h_0^0 \cup h_0^1 \cup h_1^1 \cup h_2^1 \cup h_3^1\big),\end{aligned}$$

which is irreducible. The latter 3-manifold is homeomorphic to $\tilde{Z} \cup h_1^1 \cup h_2^1 \cup h_3^1$ by a homeomorphism which takes $(\tilde{F} \cup E_0) \times \{1\}$ to $\tilde{F} \cup G_0 \cup (\partial h_0^0 - E_0)$. Therefore $\tilde{F} \cup G_0 \cup \overline{(\partial h_0^0 - E_0)}$ is incompressible in the irreducible 3-manifold $\tilde{Z} \cup h_1^1 \cup h_2^1 \cup h_3^1$.

Thus $\partial D = \partial D'$, where D' is a disk in $\tilde{F} \cup G_0 \cup \overline{(\partial h_0^0 - E_0)}$. $(\partial E_0 \times \{1\}) \cap D'$ separates D' into two subdisks, one of which lies in $\tilde{F}$ and is bounded by the union of this arc and $\tilde{\beta}$. Therefore $\tilde{\beta}$ is boundary-parallel in $\tilde{F}$, a contradiction. This completes the proof.

The proof of the following lemma is straightforward and therefore is left to the reader. A *spanning arc* in an annulus A is a properly embedded, nonboundary parallel arc.

6.6 LEMMA. *Let M be a compact, orientable, irreducible 3-manifold and F an incompressible surface in ∂M such that $\overline{(\partial M - F)}$ is incompressible. Let $D_1 \cdots D_n$ be a disjoint collection of disks in M whose boundaries are in general position with respect to ∂F. Let $D = D_1 \cup \cdots \cup D_n$. Let A be an incompressible annulus in M which is in general position with respect to D and such that $\partial A \cap \partial F = \varnothing$. If, among all such annuli isotopic to A rel ∂F, $A \cap D$ has a minimal number of components, then $A \cap D$ consists of at most spanning arcs of A.*

Now let ξ_0, ξ_1, and ξ_2 be disjoint, parallel arcs in $\mathrm{Int}((\tilde{R}_0 \cup h_0^1 \cup \tilde{R}_1) \cap \partial N)$ which join ∂E_0 and ∂E_1 and have connected intersection with h_0^1. Number so that the $\xi_i \cap h_0^1$ can be joined to the $C_i^2 \cap h_0^1$ by disjoint disks D_i in h_0^1, $i = 0, 1, 2$. Let $\tilde{C}_i^2 = C_i^2 \cup D_i \cup (\xi_i \times [0,1])$. Let $\tilde{h}_i^2$ be a regular neighborhood of $\tilde{C}_i^2$ in $\tilde{Z}$, chosen so that a product structure $\tilde{h}_i^2 = \tilde{C}_i^2 \times [-1, +1]$ is compatible with the product structure $h_i^2 = C_i^2 \times [-1, +1]$. The $\tilde{h}_i^2$ are assumed disjoint. For $i \geqslant 3$, let $\tilde{h}_i^2 = h_i^2$ and $\tilde{C}_i^2 = C_i^2$.

6.7 LEMMA. $(\tilde{Z}, \tilde{R})$ *has Property* B$'$.

PROOF. Let G be an incompressible surface in $\dot{\tilde{Z}}$ such that either $\partial G = \varnothing$ or $\partial G \subseteq (\partial M - F) \cup (\tilde{F} \cup G_0 \cup G_1)$. We may assume that $\partial G \cap (G_0 \cup G_1) = \varnothing$ and that among all surfaces in $\tilde{Z}$ isotopic to G rel $\partial \tilde{F}$, $G \cap ((H'' \cup h_0^1) \cap \tilde{C})$ has a minimal number of components. Since $(H'' \cup h_0^1) \cap \tilde{C}$ is a disjoint set of disks, G lies in $\tilde{C}$, and so, by Corollary 3.2 of [**14**], is parallel to a surface in $\tilde{C} \cap \partial M$. In particular every incompressible torus and every incompressible annulus with boundary in $\overline{(\partial \tilde{Z} - \tilde{R} - \tilde{S})}$ is boundary-parallel.

Suppose A is an incompressible annulus in $\tilde{Z}$ with boundary components α and β such that ∂A is not contained in $\overline{(\partial \tilde{Z} - \tilde{R} - \tilde{S})}$. We may assume that $A \cap \tilde{S} = \varnothing$, that A and $\cup \tilde{C}_i^2$ are in general position, and that $A \cap (\cup \tilde{C}_i^2)$ has a minimal number of components.

Case 1. *$\alpha \subseteq \tilde{R}_i$ and $\beta \subseteq \tilde{F}$.* Suppose $i \neq 0$ or 1. Then at least two 2-handles, h_k^2 and h_l^2, meet A and a common 1-handle h_j^1. At most one of these, say h_k^2, meets h_0^1. $A \cap \tilde{C}_k^2$ contains a spanning arc γ running from R_i to $\tilde{F}$. $A \cap C_l^2$ also contains a spanning arc δ running from R_i to $\tilde{F}$, but this is impossible because h_l^2 does not meet h_0^1.

Suppose $i = 0$ or 1, say $i = 0$. If α is parallel to ∂E_0 in $\tilde{R}_0$, then A can be isotoped so that $\partial A \subseteq \tilde{F}$, and therefore A is boundary-parallel. If α is parallel in $\tilde{R}_0$ to $\partial(h_j^1 \cap h_i^0)$ for some $j \neq 0$, then it meets the three 2-handles incident with h_j^1. At most one of these meets h_0^1; let h_k^2 be one which does not. Then $A \cap C_k^2$ contains a spanning arc γ joining $\tilde{R}_0$ with $\tilde{F}$, but this is impossible since h_k^2 does not meet h_0^1. If α is not boundary-parallel in $\tilde{R}_0$, then it meets at least two 2-handles which meet a common 1-handle other than h_0^1. One of these, say h_k^2, does not meet h_0^1. But $A \cap C_k^2$ contains a spanning arc γ joining $\tilde{R}_0$ to $\tilde{F}$. Again, this is impossible because h_k^2 does not meet h_0^1.

Case 2. *$\alpha \subseteq \tilde{R}_i$ and $\beta \subseteq \tilde{R}_m$.* If $i = m$, then A is $\tilde{R}_i$-compressible and thus $\tilde{R}_i$-parallel, because A intersects some C_k^2 in a spanning, R_i-parallel arc. If either α or β is parallel in $\tilde{R}$ to some ∂E_i, then A can be isotoped so that one component of ∂A is in $\tilde{R}$ and the other in $\tilde{F}$. Thus A is boundary-parallel by the previous case. We therefore assume that neither of these conditions occurs.

Subcase (a). α is boundary-parallel in $\tilde{R}_i$. Then α is parallel to $\partial(h_i^0 \cap h_j^1)$ for some $j \neq 0$. α meets all three 2-handles incident with h_j^1. At most one of these meets h_0^1. Let h_k^2 and h_l^2 be the others. $A \cap C_k^2$ contains a spanning arc γ which joins $\tilde{R}_i$ to $\tilde{R}_m$. $A \cap C_l^2$ contains a similar arc δ. It follows that h_j^1 joins h_i^0 and h_m^0. If β is not parallel in $\tilde{R}_m$ to $\partial(h_m^0 \cap h_j^1)$, then β meets some $\tilde{h}_n^2$ which does not meet h_j^1. $A \cap \tilde{C}_n^2$ contains a spanning arc ε joining $\tilde{R}_m$ to $\tilde{R}_i$. But since $\tilde{h}_n^2$ does not meet h_j^1, neither does h_n^2. But this is impossible since it meets both h_i^0 and h_m^0. Therefore β is parallel in $\tilde{R}_m$ to $\partial(h_m^0 \cap h_j^1)$. Thus A is S_j'-compressible, where S_j' is the union of S_j with the boundary-parallelisms of α and β in $\tilde{R}_i$ and $\tilde{R}_m$, respectively. Thus A is boundary-parallel.

Subcase (b). α is not boundary-parallel in $\tilde{R}_i$. Then α meets at least four distinct $\tilde{h}_n^2$. Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be arcs in the intersection of A with the corresponding $\tilde{C}_n^2$. Each γ_q joins $\tilde{R}_i$ to $\tilde{R}_m$, so each of the $\tilde{h}_n^2$ meets both h_i^0 and h_m^0. Therefore each of the h_n^2 meets both h_i^0 and h_m^0. Thus h_i^0 and h_m^0 are joined by a 1-handle h_j^1 which meets each of the four distinct h_n^2. This contradicts the fact that each 1-handle meets exactly three distinct 2-handles.

PROOF OF PROPOSITION 6.1. Let $(\lambda_{1,i}, \lambda_{2,i})$ be a copy of the true lover's tangle in a 3-cell B_i. Let Q_i be the associated tangle space. For $i \geqslant 2$, identify each 0-handle h_i^0 with B_i in such a way that $\partial(\lambda_{1,i} \cup \lambda_{2,i})$ is identified with the intersection of h_i^0 and the cores C_j^1 of the four 1-handles meeting h_i^0. For $i = 0, 1$, identify h_i^0 with B_i in such a way that the three points of $\partial(\lambda_{1,i} \cup \lambda_{2,i})$ are identified with the intersections of h_i^0 with the cores C_j^1 of the three 1-handles other than h_0^1 which meet h_i^0. Identify the remaining point of $\partial(\lambda_{1,i} \cup \lambda_{2,i})$ with $h_i^0 \cap \mu_i$. Perform all these identifications in such a manner that

$$\tilde{J} = \bigcup_i (\lambda_{1,i} \cup \lambda_{2,i}) \underset{j \neq 0}{\cup} C_j^1 \cup \mu_0 \cup \mu_1$$

is a single arc.

Let $Q = \bigcup_i Q_i$. Then the exterior $\tilde{X}$ of $\tilde{J}$ is $Q \cup \tilde{Z}$. Note that $Q \cap \tilde{Z} = \tilde{R}$. By Proposition 4.1 Q is a simple Haken manifold. Since $\tilde{R}$ and $\overline{(\partial Q - \tilde{R})}$ are incompressible in Q, $(Q, \tilde{R})$ has Property C′. By Lemma 6.7 $(\tilde{Z}, \tilde{R})$ has Property B′. Thus by Lemma 2.5 $\tilde{X}$ is a simple Haken manifold.

6.8. QUESTION. *Let M be as in Proposition* 6.1. *If ∂M contains two different components F_0 and F_1, does there exist a properly embedded arc $\tilde{J}$ in M joining F_0 and F_1 whose exterior is a simple Haken manifold*?

7. Relative homology cobordisms and link concordances.

7.1 THEOREM. *Let M_0^3 be a compact, connected, oriented* 3-*manifold such that $\partial M^3 \neq \varnothing$ and contains no* 2-*spheres. Then M_0^3 is relative homology cobordant to a hyperbolic* 3-*manifold M_1^3.*

7.2 THEOREM. *Let M^3 be a compact, connected, oriented* 3-*manifold such that ∂M^3 contains no* 2-*spheres. Then every link L_0 in M^3 is concordant to a link L_1 with hyperbolic exterior.*

7.3 COROLLARY. *Every knot K_0 in S^3 is concordant to a knot K_1 with hyperbolic exterior.*

7.4 COROLLARY (KIRBY AND LICKORISH [7], LIVINGSTON [**8**]). *Every knot K_0 in S^3 is concordant to a prime knot K_1.*

We shall give a unified proof of these theorems by means of

7.5 PROPOSITION. *Let X^3 be a compact, connected, oriented* 3-*manifold such that ∂X^3 contains no* 2-*spheres. Then every wedge of circles G_0 in Int X^3 is concordant to a wedge of circles G_1 in Int X^3 whose exterior is hyperbolic.*

PROOF. Let Y^3 be the exterior of G_0 in X^3. Let F^2 be the component of ∂Y^3 which bounds $H^3 = \overline{(X^3 - Y^3)}$. Let μ_1' and μ_2' be disjoint arcs in the same circle of G_0, chosen disjoint from the wedge point. Let E_1^3 and E_2^3 be disjoint regular neighborhoods of μ_1' and μ_2' which meet H^3 in annuli A_1^2 and A_2^2, respectively. By Proposition 6.1 there is a properly embedded arc $\tilde{J}$ in Y^3 whose exterior W^3 is a simple Haken manifold and such that $\partial \tilde{J} \subseteq F^2$. Let $Z^3 = \overline{(Y^3 - W^3)}$. We may assume that $Z^3 \cap H^3$ consists of one disk in each of A_1^2 and A_2^2.

Let $\{\mu_1, \mu_2\}$ be the composition of the true lover's tangle $\{\lambda_1, \lambda_2\}$ with its reflection in a plane, as in §5. Identify B^3 with $E_1^3 \cup Z^3 \cup E_2^3$ in such a way that $\partial\mu_1 = \partial\mu_1'$ and $\partial\mu_2 = \partial\mu_2'$. Let $G_1 = (G_0 - \mu_1' - \mu_2') \cup \mu_1 \cup \mu_2$. $\{\mu_1, \mu_2\}$ is concordant in B^3 to the trivial tangle $\{\mu_1', \mu_2'\}$. Therefore G_0 and G_1 are concordant in X^3.

Let Q^3 be the tangle space of $\{\mu_1, \mu_2\}$. $Q^3 \cap W^3$ is a 2-sphere with four holes which is an incompressible subsurface of both ∂Q^3 and ∂W^3. Q^3 is a simple Haken manifold by Proposition 4.1. W^3 is a simple Haken manifold by Proposition 6.1. Thus both $(Q^3, Q^3 \cap W^3)$ and $(W^3, Q^3 \cap W^3)$ have Property C′. Therefore by Lemma 2.5, the exterior $Q^3 \cup W^3$ of G_1 is a simple Haken manifold and thus is hyperbolic.

Proof of 7.1. Let F^2 be a component of ∂M_0^3. Let H^3 be a cube with handles such that ∂H^3 is homeomorhic to F^2. Construct X^3 by attaching H^3 to M_0^3 via this homeomorphism. Let G_0 be a wedge of circles of which H^3 is a regular neighborhood in X^3. By Proposition 7.5 G_0 is concordant to G_1 such that the exterior M_1^3 of G_1 in X^3 is hyperbolic. By Lemma 2.7 the exterior $(P^4; M_0^3, M_1^3)$ of this concordance is a relative homology cobordism.

Proof of 7.2. Let $K_1, \ldots, K_n$ be the components of L_0, with disjoint regular neighborhoods $U_1, \ldots, U_n$ in M^3. Let $X^3 = \overline{(M^3 - (U_2 \cup \cdots \cup U_n))}$ and $G_0 = K_1$. By Proposition 7.5 G_0 is concordant in X^3 to G_1 such that the exterior of G_1 in X^3 is hyperbolic. By taking product concordances $K_i \times [0, 1]$ in $U_i \times [0, 1]$ for $i \geqslant 2$, we see that $L_0 = \{G_0, K_2, \ldots, K_n\}$ is concordant in M^3 to $L_1 = \{G_1, K_2, \ldots, K_n\}$. The exterior of L_1 in M^3 is the same as that of G_1 in X^3 and so is hyperbolic.

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